Ring Theory 4

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Q) If $a \neq b$, $a, b \in R$ and $a^3 = b^3$ and $a^2b = b^2a$. Show that a^2+b^2 is not a unit.

Ans:- $a, b \in R$

$a - b \neq 0$

$(a^2+b^2)(a-b) = a^3 - a^2b + b^2a - b^3 = a^3 - b^3 + a^2b - b^2a$

$= 0$

$a - b \neq 0$

So not unit

Q) Center of $M_2(R)$

Ans:- $x \in M_2(R)$

$C = \{ c \in M_2(R) \mid cx = xc \ \forall x \in M_2(R) \}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$C = \{ 1, -1 \}$

$$\Rightarrow \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} = \begin{bmatrix} wa+xc & wb+xd \\ ya+zc & yb+zd \end{bmatrix}$$

Ring Homomorphism :-

R_1 and R_2 be two rings.

Def:- A ring homomorphism is a map $\phi : R_1 \longrightarrow R_2$ which satisfies,

(i) $\phi(a+b) = \phi(a) + \phi(b) \quad \forall a, b \in R_1$

(ii) $\phi(ab) = \phi(a)\phi(b) \quad \forall a, b \in R_1$

•> $\ker(\phi) = \{ a \mid \phi(a) = 0 \}$

Def:- A bijective ring homomorphism is called an isomorphism

Def:- A isomorphism from $R \to R$ is an automorphism

•> Unit (Multiplicative identity) is preserved, ie,

$$\phi(1_{R_1}) = 1_{R_2}$$

•> If ϕ is a bijective homomorphism then its inverse, ϕ^{-1}, is also a ring homomorphism

•> This is a kind of extension of group homomorphism.

•> $\phi(0_{R_1}) = 0_{R_2}$

•> $\phi(-a) = -\phi(a) \;\; \forall a \in R_1$

•> If a is a unit then $\phi(a^{-1}) = \phi(a)^{-1}$

•> Composition of two ring homomorphism is also a ring homomorphism

$$\phi_1 : R_1 \to R_2, \quad \phi_2 : R_2 \to R_3$$

Then, $\phi_1 \circ \phi_2 : R_1 \to R_3$ is also Ring homomorphism

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•> $\phi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$

$$a \longrightarrow a \pmod{n}$$

It is a surjection and a ring homomorphism

$\ker(\phi) = \{ a \mid \phi(a) = 0 \} = n\mathbb{Z}$

• $\varphi_n : \mathbb{Z} \to \mathbb{Z}$
$x \to nx$

It is a ring homomorphism iff $n = 1$.
$\mathrm{Ker}(\varphi_1) = 0$

Proposition: — Let R and S be rings and $\varphi : R \to S$ be a homomorphism

(1) The image of φ is a subring of S

(2) The kernel of φ is a subring of R. Also, if $\alpha \in \mathrm{Ker}(\varphi)$ then $r\alpha, \alpha r \in \mathrm{Ker}(\varphi)$ $\forall r \in R$.

Proof: — (1) $a, b \in \mathrm{Im}(\varphi)$

$\exists r_1, r_2 \in R$ such that, $\varphi(r_1) = a$ & $\varphi(r_2) = b$

$a - b = \varphi(r_1) - \varphi(r_2) = \varphi(r_1 - r_2) \in \mathrm{Im}(\varphi)$
as $r_1 - r_2 \in R$

$ab = \varphi(r_1)\varphi(r_2) = \varphi(r_1 r_2) \in \mathrm{Im}(\varphi)$

So $\mathrm{Im}(\varphi)$ is a subring

(2) $a, b \in \mathrm{Ker}(\varphi) \Rightarrow \varphi(a) = \varphi(b) = 0$

$\varphi(a-b) = \varphi(a) - \varphi(b) = 0 \Rightarrow a - b \in \mathrm{Ker}(\varphi)$

$\varphi(ab) = \varphi(a)\varphi(b) = 0 \Rightarrow ab \in \mathrm{Ker}(\varphi)$

So $\mathrm{Ker}(\varphi)$ is a subring

$\varphi(r\alpha) = \varphi(r)\varphi(\alpha) = 0 = \varphi(\alpha)\varphi(r) = \varphi(\alpha r)$
$\Rightarrow r\alpha, \alpha r \in \mathrm{Ker}(\varphi)$

Definition: — Let R be a ring and let I be a subset of R and let $r \in R$.

(1) $rI = \{ra \mid a \in I\}$ and $Ir = \{ar \mid a \in I\}$

$I \subseteq R$ is a left ideal of R if

(1) $rI = \{ra \mid a \in I\}$

(2) A subset I of R is a left ideal of R if

(i) I is a subring of R

(ii) I is closed under left multiplication by elements from R, i.e., $rI \subseteq I \ \forall r \in R$

Similarly for right ideal

(3) A subset I is ideal if it is both left and right ideal of R

- $(I, +)$ is a subgroup of $(R, +)$
- For every $r \in R$ and every $x \in I$ we get rx in I

- Even integers of ring $\mathbb{Z}$ is an ideal.
- Set of all polynomials with real coefficients which are divisible by the polynomial x^2+1 is an ideal of the ring $\mathbb{R}[x]$

<u>Proposition</u>:- Let R be a ring and I be an ideal of R. Then the (additive) quotient group R/I is a ring under binary operation.

$$\underset{\in R/I}{(r+I)} + \underset{\in R/I}{(s+I)} = \underset{\in R/I}{(r+s+I)} \quad \text{and} \quad \underset{\in R/I}{(r+I)} \times \underset{\in R/I}{(s+I)} = \underset{\in R/I}{(rs)+I}$$

$\forall r, s \in R$.

- Also, the converse is true, i.e., if I is any subgroup such that the above operations are well defined then I is an ideal of R

<u>Def</u>:- When I is an ideal of R then the ring R/I with the binary operations as in above proposition is called the ... R by I.

Def: W... is called the
the binary operations as in above proposition
quotient ring R by I.